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ACOUSTIC SPEEDS FOR STRATIFIED TWO-PHASE FLUIDS IN RECTANGULAR DUCTS

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Abstract—A two-dimensional, time harmonic, free-mode analysis has been applied to the stratified regime, assuming a planar interface at constant height separating two fluids. The resulting dispersion relation has been solved asymptotically in the common case of one fluid being much lighter than the other, and a set of acoustic modes, apparently unnoticed previously, has been highlighted.

1. INTRODUCTION

This study was prompted by a recent publication by Nguyen *et al.* (1981), (hereafter referred to as NWG), in which a simple but general theory was developed for several two-phase flow regimes, including stratified, slug and homogeneous flows. That theory was based on the notion that the interface between the two phases behaves elastically in a similar manner to an elastic tube (Wylie & Streeter 1978). However, the widely used formulae for sound speeds in elastic tubes are founded on a quasi-steady formulation in which flexural wall waves are ignored; in acoustic terms, the walls are regarded as locally-reacting and are represented by an impedance condition. Clearly, such a model is physically unrealistic, and inclusion of flexural motion produces more meaningful results (Sinai 1981).

A similar difficulty arises in the two-phase situation, since the interface separates two wave-bearing media, and questions therefore arise as to the veracity of NWG's analysis. In fact, the calculations described in the present paper indicate that the NWG theory violates the kinematic and dynamic conditions at the interface, namely equality of displacement and pressure on both sides of that discontinuity. Indeed, NWG confined themselves to one-dimensional motion, so that their picture of wave propagation in horizontal stratified regimes consisted of vertical wave fronts in both the liquid and the gas, travelling at different speeds: this implies discontinuous pressures.

For these reasons, a rigorous analysis is presented herein, within the confines of linear theory, and for the sake of mathematical tractability attention is focused on the two-dimensional stratified regime ignoring mean, background flow as well as gravitational and surface tension effects. Moreover, the motion is assumed to be time-harmonic, so that the "free-mode" analysis produces a dispersion relation for the axial wave number as a function of frequency. The dispersion relation cannot be solved analytically in general, so that sometimes it may be necessary to solve it numerically. However, the present paper is confined to the common situation in which the density of one fluid is much smaller than that of the other, and it is then possible to obtain explicit asymptotic results in terms of the ratio of the two densities.

Morioka & Matsui (1975) appear to have been the first to carry out a two-dimensional analysis, and a number of papers have since compared two-dimensional and one-dimensional theories (e.g. Banerjee & Chan 1980; Ardron 1980; Wallis & Hutchings 1983). The present contribution identifies a set of acoustic modes which do not seem to have been noticed by Morioka & Matsui and other workers.

The interesting point in NWG's paper is the good agreement between theory and experiment. However, for the stratified regime their theory yielded two formulae, one predicting a phase speed close to that of the gas and the other predicting a speed much lower than that speed. They found that the former agreed well with experiment (Henry *et al.* 1971) (hereafter referred to as HGF), whereas the latter did not. The present paper explains these contradictions and quantifies the phase speeds of the higher "duct" modes precluded from NWG's analysis.

It should be emphasized that this paper does not present the results for a particular boundary or initial-value problem, and in applying the predictions of a time-harmonic analysis to the evolution of a particular pulse, one should assess the frequency spectrum and utilize the known techniques for wave propagation in dispersive media (Whitham, 1974). However, under certain conditions the phase speeds are virtually independent of frequency, so that within the context of the current assumption the pulse form will remain unaltered as it propagates.

The assumptions made are the following:

(i) the duct walls are rigid

(ii) the void fraction is constant and no large waves exist on the interface

(iii) fluid viscosity, conduction and phase changes are ignored

(iv) if fluid motion arises, the Mach number is small

(v) all disturbances are small with respect to the background state, thereby permitting the use of linear theory

(vi) the motion is two-dimensional in space.

2. THE GENERAL DISPERSION RELATION

Referring to figure 1, the velocity potential $\tilde{\phi}(x, y, t)$ satisfies the wave equation in both fluids:

region I:

$$c_1^2(\tilde{\phi}_{xx} + \tilde{\phi}_{yy}) - \tilde{\phi}_{tt} = 0 \qquad [2.1a]$$

region II:

$$c_2^2(\tilde{\phi}_{xx} + \tilde{\phi}_{yy}) - \tilde{\phi}_{tt} = 0 \qquad [2.1b]$$

where c_1 and c_2 are the acoustic sound speeds in regions I and II, respectively. The search for time-harmonic waves travelling in the axial direction is implemented by decomposing $\tilde{\phi}$ into the form

$$\tilde{\phi}(x, y, t) = e^{i(kx - \omega t)}\phi(y). \qquad [2.2]$$

Here k is the axial wave number for the two-fluid system, ω is the frequency (rad.s^{-1}) and the ϕ's for the two fluids will be different.

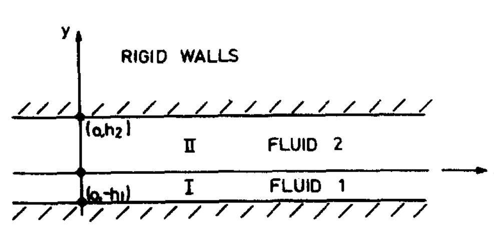

Figure 1. The stratified configuration.

The general solutions of [2.1] are:

$$\phi_{\rm I} = A e^{\lambda_1 y} + B e^{-\lambda_1 y} \qquad [2.3a]$$

$$\phi_{\rm II} = C e^{\lambda_2 y} + D e^{-\lambda_2 y} \qquad [2.3b]$$

where

$$\lambda_j = (k^2 - K_j^2)^{1/2}, \quad K_j = \omega/c_j, \quad j = 1, \text{or } 2. \qquad [2.4]$$

It should be mentioned that since waves may travel in both the negative and positive-y directions, the precise interpretation of λ_j when both k and K_j are real and $k < K_j$ is unimportant. Also, the velocity potential is defined in such a way that the perturbation velocity vector and pressure are given by $\nabla\tilde{\phi}$ and $-\rho_j\partial\tilde{\phi}/\partial t$, respectively, where ρ denotes density.

Applying the boundary and interfacial conditions leads to the dispersion relation (Morioka & Matsui 1975)

$$\lambda_2 \tanh(\lambda_2 h_2) + \rho\lambda_1 \tanh(\lambda_1 h_1) = 0; \quad \rho = \rho_2/\rho_1. \qquad [2.5]$$

A check on this situation is provided by the various possible limits:

(i) $$h_2 = 0, \quad \lambda_1 \sinh(\lambda_1 h_1) = 0$$

viz.

$$\lambda_1 h_1 = \pm in\pi, \quad n = 0, 1, 2, \ldots \qquad [2.6]$$

or

$$k^2 = K_1^2 - \left(\frac{n\pi}{h_1}\right)^2$$

which is the classical result for single-phase fluid (Rayleigh 1945). Associated with [2.6] is the notion of a cut-off frequency ω_n below which k becomes pure-imaginary, implying a rapid damping of the particular mode:

$$\omega_n = n\pi c_1/h_1. \qquad [2.7]$$

A similar limit arises when $h_1 \to 0$.

(ii) $\rho = 1$, $\lambda_1 = \lambda_2 = \lambda$: this corresponds to a single-phase fluid also and (2.5) becomes

$$\sinh[\lambda(h_1 + h_2)] = 0$$

which is the correct limit.

In conclusion, it should be emphasized that a value of λ_j which is real corresponds to plane waves in the particular fluid, and imaginary λ_j represents oblique waves associated with higher duct modes; the unfamiliar reader is referred to Rayleigh (1945) and Morse & Ingard (1968). The textbooks also discuss the transmission of oblique acoustic waves across material interfaces, and it is useful to remember that the character of the field in a fluid depends on the speed of propagation of the disturbance at its boundary; if the speed is supersonic, oblique waves arise, but if the speed is subsonic an exponentially-decaying surface wave exists, with "wavefronts" perpendicular to the boundary (this corresponds to total internal reflection).

In the current problem the situation is complicated a little by the presence of the two rigid surfaces, but all these phenomena are encapsulated in the form of [2.3] and the solution of the dispersion relation.

3. ASYMPTOTIC RESULTS FOR SMALL DENSITY RATIO

As discussed in section 1, it is not possible to solve [2.5] analytically for k as a function of ω, but the quantity ρ is frequently very small, and it is then a straightforward matter to derive asymptotic results. Write the relation [2.5] as:

$$F = f_0(k) + \rho f_1(k) = 0 \qquad [3.1a]$$

where

$$f_0 = \lambda_2 \cosh(\lambda_1 h_1) \sinh(\lambda_2 h_2) \qquad [3.1b]$$

$$f_1 = \lambda_1 \sinh(\lambda_1 h_1) \cosh(\lambda_2 h_2). \qquad [3.1c]$$

In a first approximation, k is given by $k^{(0)}$,where:

$$f_0[k^{(0)}] = 0. \qquad [3.2]$$

It follows that the $0(\rho)$ correction to k is given by:

$$k = k^{(0)} + \rho k^{(1)}; \quad k^{(1)} = -(f_1/f_0') \text{ evaluated at } k = k^{(0)} \qquad [3.3]$$

provided $f_0'[k^{(0)}]$ exists. Here $f_0' - \partial f_0/\partial k$.

Actually, when dealing with plane modes, it turns out that f_0' does not exist, and it is then necessary to expand f_0 and f_1 formally, term by term.

As may be seen from [3.1b], two sets of solutions exist: those associated with the vanishing of $\cosh(\lambda_1 h_1)$, referred to as "heavier-fluid modes" and those associated with the vanishing of $\sinh(\lambda_2 h_2)$, referred to as "lighter-fluid modes". The two sets will be considered separately. The heavier-fluid modes do not appear to have been noticed previously. Morioka & Matsui (1975) did solve the dispersion relation numerically, but as an initial guess their numerical algorithm utilised the asymptotic approximation to the "ligher-fluid" modes, and it is therefore not surprising that the heavier-fluid modes were not identified numerically. The inclusion of the residues associated with these modes would modify the profiles they obtained for the temporal evolution of an initial pressure step.

3.1 *Lighter-fluid modes*

The relevant solutions of [3.2] are given by

$$\lambda_{2_n}^{(0)} h_2 = -in\pi, \quad n = 0, 1, 2, \ldots \qquad [3.1.1]$$

where λ_2 has also been decomposed into an asymptotic form:

$$\lambda_j = \lambda_j^{(0)} + \lambda_j^{(1)}, \quad \lambda^{(1)} \ll \lambda^{(0)} \qquad [3.1.2]$$

It should be noted, though, that $\lambda^{(1)}$ is not always an $0(\rho)$ correction. It is seen immediately from [3.1.1] than $n = 0$ will lead to an almost-planar mode propagating at a speed close to c_2 which is evidently the case observed by HGF.

In order to emphasize the modal forms, re-write [3.1.2] as follows:

$$\lambda_{2_n} h_2 = -in\pi + \mu_n. \tag{3.1.3}$$

From [3.1.3], the wavenumber will be deduced as:

$$k = k_n = k_n^{(0)} + k_n^{(1)}, \quad k_n^{(1)} \ll k_n^{(0)} \tag{3.1.4a}$$

where

$$k_n^{(0)} = \left[K_2^2 - \left(\frac{n\pi}{h_2}\right)^2\right]^{1/2}, \quad \omega_n \simeq \frac{n\pi c_2}{h_2}. \tag{3.1.4b}$$

Consider first the plane mode $n = 0$. Substitution of [3.1.3] in [3.1] leads to:

$$\mu_0^2 = -\rho h_2 (\lambda_1)_2 \tanh (\lambda_1 h_1)_2 \tag{3.1.5a}$$

where

$$(\lambda_1)_2 = (\lambda_1)_{k=K_2} = (K_2^2 - K_1^2)^{1/2}. \tag{3.1.5b}$$

Equations [3.1.3] and [3.1.5] imply that:

$$k_0^2 = K_2^2 - \frac{\rho}{h_2}(\lambda_1)_2 \tanh \lambda_1 h_1. \tag{3.1.6}$$

Noting that the phase speed is given by ω/k, [3.1.6] may be written as:

$$\frac{1}{a_0^2} = \frac{1}{c_2^2} - \frac{\rho}{\omega^2 h_2}(K_2^2 - K_1^2)^{1/2} \tanh [(K_2^2 - K_1^2)^{1/2} h_1]. \tag{3.1.7}$$

Evidently, this mode is slightly faster than c_2. Several simplifications may arise. First, if fluid 1 depth is "compact", viz. the frequency is sufficiently small that $|(\lambda_1)_2 h_1| \ll 1$, [3.1.7] becomes:

$$\frac{1}{a_0^2} \simeq \frac{1}{c_2^2} - \rho\left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)\left(\frac{1-\alpha}{\alpha}\right) \tag{3.1.8}$$

where the void fraction α, in the present regime, is given by:

$$\alpha = \frac{h_2}{h_1 + h_2}. \tag{3.1.9}$$

Furthermore, if $c_2 \ll c_1$, this simplifies to:

$$\left.\begin{aligned} \frac{1}{a_0^2} &\simeq \frac{1}{c_2^2}\left[1 - \rho\left(\frac{1-\alpha}{\alpha}\right)\right] \\ a_0 &\simeq c_2\left[1 + \frac{1}{2}\rho\left(\frac{1-\alpha}{\alpha}\right)\right] \end{aligned}\right\} \quad (K_2 \gg K_1). \tag{3.1.10}$$

Another interesting limit is the non-compact condition, viz. high frequencies at which $|(\lambda_1)_2 h_1| \to \infty$:

$$\frac{1}{a_0^2} \sim \frac{1}{c_2^2} - \frac{\rho}{\omega h_2}\left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right)^{1/2} \tag{3.1.11}$$

Equation [3.1.11] is the relevant limit if the imposed signal possesses significant high frequency components, such as in a Heaviside step. It is instructive to compare the present results with the more successful of the two formulae presented by NWG (in the current notation):

$$\frac{1}{a_0^2} = \frac{1}{c_2^2}\left[1 + \rho\left(\frac{1-\alpha}{\alpha}\right)\frac{c_2^2}{c_1^2}\right].$$

It is seen that NWG's small correction does not agree with any of the forms presented herein, although in the circumstances the correction is so small as to be unimportant anyway. Nevertheless, these disagreements should not be underestimated, because they reflect the possible shortcomings in the one-dimensional modelling, and NWG's formula is not expected to be valid when ρ is *not* small. Moreover, the present theory does not predict any phase speed resembling NWG's second formula (which indeed they found to be in disagreement with experiment).

It is noteworthy that several non-uniformities exist wherein the asymptotic expansion [3.3] breaks down, but these exist over narrow bands of frequency and will not be discussed here.

Turning now to the higher modes $n > 0$, substitution of [3.1.3] in [3.1] yields, after retention of $0(\rho)$ terms

$$\mu_n = -i\left(\frac{\rho h_2}{n\pi}\right)\lambda_{1_n} \tanh(\lambda_{1_n} h_1) \qquad [3.1.12a]$$

where

$$\lambda_{1_n} = (\lambda_1)_{k=k_n^{(0)}} = [K_2^2 - (n\pi/h_2)^2 - K_1^2]^{1/2} = [(\lambda_1)_2^2 - (n\pi/h_2)^2]^{1/2}. \qquad [3.1.12b]$$

Above cut-off, λ_{1_n} will be real if $(\lambda_1)_2 > n\pi/h_2$ and imaginary if $(\lambda_1)_2 < n\pi/h_2$.

Equation [3.1.12a] implies that:

$$k_n = k_n^{(0)} - \frac{\rho}{h_2 k_n^{(0)}}\lambda_{1_n} \tanh(\lambda_{1_n} h_1) \qquad [3.1.13]$$

or

$$\frac{1}{a_n} = -\frac{1}{a_n^{(0)}} - \frac{a_n^{(0)}}{\omega^2 h_2^2}\lambda_{1_n} \tanh(\lambda_{1_n} h_1) \qquad [3.1.14]$$

where

$$\frac{1}{a_n^{(0)}} = \left[\frac{1}{c_2^2} - \left(\frac{n\pi}{\omega h_2}\right)^2\right]^{1/2} \qquad [3.1.15]$$

a_n is the axial phase speed of the nth second-fluid duct mode, and [3.1.14] shows that $a_n \geqslant a_n^{(0)}$, irrespective of the sign of $k_n^{(0)} - K_1$.

This completes the assessment of the lighter-fluid modes. Which of these modes will appear in a particular situation will obviously depend on the method used to excite the system.

3.2 *Heavier-fluid modes*

These modes are associated with the condition

$$\cosh(\lambda_1 h_1) = 0 \qquad [3.2.1a]$$

viz.

$$\lambda_{1_m}^{(0)} h_1 = -iq\pi, \quad q = m + 1/2, \quad m = 0, 1, 2, \ldots \qquad [3.2.1b]$$

As before

$$\lambda_m h_1 = -iq\pi + \zeta_m, \quad k_m = k_m^{(0)} + k_m^{(1)} \qquad [3.2.2]$$

where

$$k_m^{(0)} = [K_1^2 - (q\pi/h_1)^2]^{1/2}. \qquad [3.2.3]$$

The cut-off frequencies are given approximately by:

$$\omega_m \simeq q\pi c_1/h_1 \qquad [3.2.4]$$

and it is noteworthy that the lowest mode $m = 0$, is non-planar in I and possesses a non-vanishing cut-off frequency. Thus, if $c_1 < c_2$ plane modes could exist in II, but if $c_1 > c_2$ no plane modes can exist at all under the present circumstances.

Substituting in [3.1] as before

$$\zeta_m = \frac{iq\rho\pi}{h_1\lambda_{2_m}} \coth(\lambda_{2_m} h_2) \qquad [3.2.5a]$$

where

$$\lambda_{2_m} = [k_m^{(0)2} - K_2^2]^{1/2}. \qquad [3.2.5b]$$

Hence

$$k_m = k_m^{(0)} + -\frac{\rho}{k_m^{(0)} h_1 \lambda_{2_m}} \left(\frac{q\pi}{h_1}\right)^2 \coth(\lambda_{2_m} h_2) \qquad [3.2.6]$$

$$\frac{1}{a_m} = -\frac{1}{a_m^{(0)}} + \frac{\rho a_m^{(0)}}{\omega^2 h_1 \lambda_{2_m}} \left(\frac{q\pi}{h_1}\right)^2 \coth(\lambda_{2_m} h_2) \qquad [3.2.7]$$

where

$$\frac{1}{a_m^{(0)}} = \left[\frac{1}{a_1^2} - \left(\frac{q\pi}{\omega h_1}\right)^2\right]^{1/2}. \qquad [3.2.8]$$

If

$$a_2 \ll a_1, \quad \lambda_{2_m} \simeq -iK_2 \qquad [3.2.9a]$$

and [3.2.7] becomes

$$\frac{1}{a_m} \simeq \frac{1}{a_m^{(0)}} - \frac{\rho a_m^{(0)}}{\omega^2 K_2 h_1} \left(\frac{q\pi}{h_1}\right)^2 \cot(K_2 h_2). \qquad [3.2.9b]$$

Several non-uniformities arise here too, for example $K_2 h_2 = m\pi$, but their discussion will be postponed for the moment.

If $K_2h_2 \ll 1$, [3.2.9b] becomes

$$\frac{1}{a_m} \approx \frac{1}{a_m^{(0)}}\left[1 - \frac{a_m^{(0)2}\rho}{\alpha(1-\alpha)^3H^4}\left(\frac{q\pi}{\omega K_2}\right)^2\right]. \qquad [3.2.10]$$

Evidently, this mode is slightly faster than $a_m^{(0)}$.

4. CONCLUSION

A free-mode time-harmonic analysis has yielded the familiar dispersion relation for acoustic waves propagating in stratified two-phase or two-fluid system.

Explicit asymptotic solutions of this relation have been obtained for the frequent situations in which the density of one fluid is much smaller than that of the other.

If the heavier fluid also possesses a sound speed which is larger than that of the lighter fluid (see figure 1), the results maybe summarised as follows. One planar mode (k_0) propagates at a speed close to c_2 with amplitude remaining virtually constant in II but varying with y in I. Additional modes (k_n) belonging to the same set are faster than c_2, are invariably non-planar in II and may or may not be planar in I (depending on whether the phase speed is less or greater than c_1). An additional set of modes (k_m) is faster than c_1 and is invariably non-planar in both I and II.

The results explain the good agreement between one of NWG's formulae and the experiments, and identify a set of acoustic modes which have apparently remained unnoticed previously.

NOMENCLATURE

- a_n duct acoustic phase speed associated with nth mode
- c_1 acoustic speed in fluid I
- c_2 acoustic speed in fluid II
- h_1, h_2 fluid layer thickness, see figure 1
- H total duct height
- k axial wavenumber, viz. frequency/axial phase speed
- K_1 ω/c_1
- K_2 ω/c_2
- x, y Cartesian co-ordinates, see figure 1
- t time

Greek symbols

- α void fraction in [3.1.9]
- λ_1 $(k^2 - K_1^2)^{1/2}$
- λ_2 $(k^2 - K_2^2)^{1/2}$
- ρ_1 density of fluid I
- ρ_2 density of fluid II
- ρ ρ_2/ρ_1
- ϕ velocity potential
- ω radial frequency, rads. s^{-1}

REFERENCES

ARDRON, K. H. 1980 One-dimensional two-fluid equations for horizontal stratified two-phase flow. *Int. J. Multiphase Flow* **6**, 295–304.

BANERJEE, S. & CHAN, A. M. C. 1980 Separated flow models—I. Analysis of the averaged and local instantaneous formulations. *Int. J. Multiphase Flow* **6**, 1–24.

HENRY, R. E., GROLMES, M. A. & FAUSKE, H. K. 1971 Pressure pulse propagation in two-phase one—and two—component mixtures. *Argonne National Laboratory Report** 7792.

MORIOKA, S. & MATSUI, G. 1975 Pressure-wave propagation through a separated gas–liquid layer in a duct. *J. Fluid Mechanics* **70**, 721–731.

MORSE, P. M. & INGARD, K. U. 1968 *Theoretical Acoustics*. McGraw-Hill, New York.

NGUYEN, D. L., WINTER, E. R. F. & GREINER, M. 1981 Sonic velocity in two-phase systems. *Int. J. Multiphase Flow* **7**, 311–320.

RAYLEIGH, J. W. S. 1945 *The Theory of Sound.* Dover, New York.

SINAI, Y. L. 1981 Two-dimensional acoustic wave propagation in elastic ducts. *J. Sound Vib.* **76**, 517–528.

WALLIS, G. B. & HUTCHINGS, B. J. 1983 Compressibility effects on waves in stratified two-phase flow. *Int. J. Multiphase Flow* **9**, 325–336.

WHITHAM, G. B. 1974 *Linear and Non-linear Waves*. Wiley, New York.

WYLIE, E. B. & STREETER, V. L. 1978 *Fluid Transients*. McGraw-Hill, New York.